The Symmetry of a Tiling

July 30, 2018

1 Introduction

This website uses a new notation for the symmetry of tiling patterns defined in [1]. The conventional notation is also noted since this is widely used, as in [2]. The older notation is somewhat *ad hoc*, whereas the new notation, once understood, has a logical structure and meaning.

Those wishing to explore these issue further should consult [1] since only an outline is given here. Clicking on the figure number produces the details of the tiling in the main collection.

2 The new notation — planar groups

2.1 Mirror lines

We first consider *mirror lines*. These should be easy to see. Consider the pattern in Figure 1.

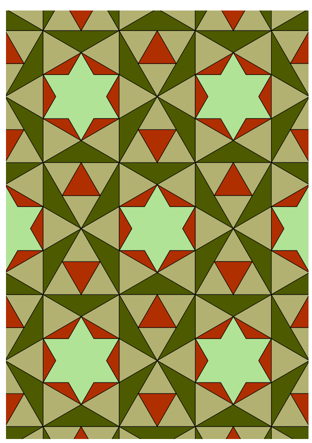

Figure 1: Example of mirror lines

The mirror lines go from one corner of the 6-pointed star to the opposite corner (and extended indefinitely in both directions).

You can see that these mirror lines divide the patterns into triangles, one vertex at the centre of the 6-pointed star, one at the centre of the red equilateral triangle and the other where the four green triangles meet.

It is easy to see that these small triangles have no internal symmetry and hence the mirror lines captures the symmetry of the pattern. Round the vertices of this triangle, we have 6-fold, 3-fold and 2-fold symmetry. The notation (signature) of this symmetry group is *632. The star indicates the use of mirror lines and the numbers the lines meeting at a point.

We now use the same logic with the pattern in Figure 2.

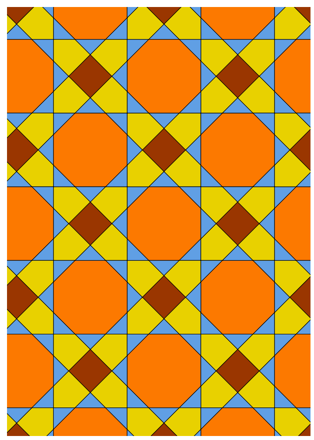

Figure 2: Second example of mirror lines

One set of mirror lines are vertical and horizontal cutting the small brown squares in two. Another set are the diagonals through the middle of the orange octagons. The triangles formed by the mirror lines have vertices at the centre of the octagon, the center of the brown square and the point where the two blue triangles meet. Round the vertices of this triangle, we have 4-fold, 4-fold and 2-fold symmetry. Hence the signature for the symmetry of this pattern is *442.

Our third example with mirror lines is Figure 3.

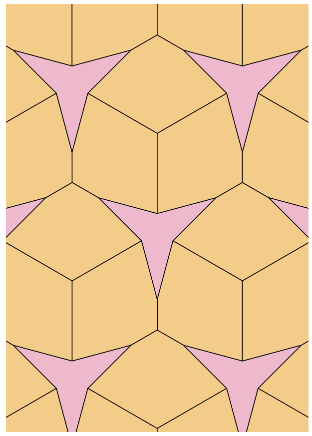

Figure 3: Third example of mirror lines

Here the mirror lines are all the lines *except* the edges of the three-pointed star. Hence the pattern is divided into triangles with vertices at the centre of the three-pointed star and the two other places where the mirror lines cross. We have 3-fold symmetry round each of these vertices, hence the signature for the symmetry of this pattern is *333.

Our fourth example with mirror lines is in Figure 4.

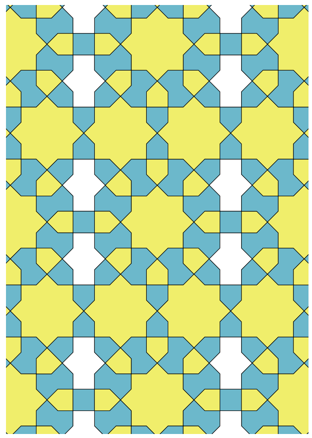

Figure 4: Fourth example of mirror lines

Here the mirror lines are vertical and horizontal. One set of vertical lines goes through the white polygons, the other through the 8-pointed star which appears between the four white polygons. The horizontal ones goes through either the the 8-points stars only or through the 8-pointed star and square. Hence the pattern is divided into rectangles with 2-fold symmetry at each vertex. Hence the signature for this pattern is *2222.

2.2 Gyrations

Here we consider points in the pattern about which a rotation can be undertaken to leave the pattern unchanged. Corresponding to each of the four pattern above, we have a pattern with the same number of gyrations. All these patterns are different from their mirror image (and have no mirror lines).

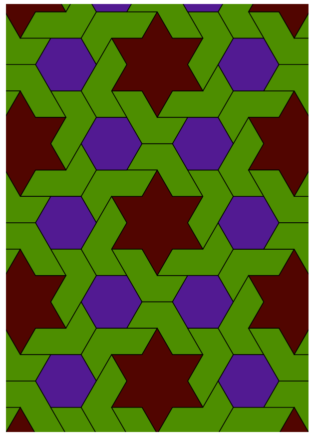

Figure 5: 632

In Figure 5, the centres of the gyrations are the center of the hexagons, centres of the six-pointed star and the point midway between two neighbouring hexagons.

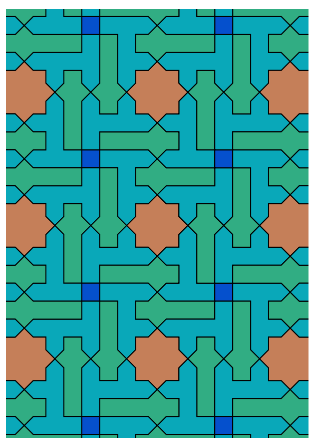

Figure 6: 442

In Figure 6, the centres of the gyrations are the center of the 8-pointed star, centres of the square and the point midway between two neighbouring 8-pointed stars.

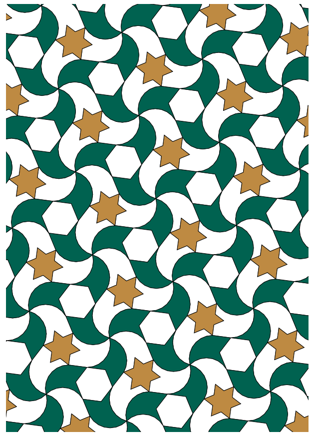

Figure 7: 333

In Figure 7, the centres of gyration at at the centre of the six-pointed star, centre of the white hexagon and where three green polygons meet.

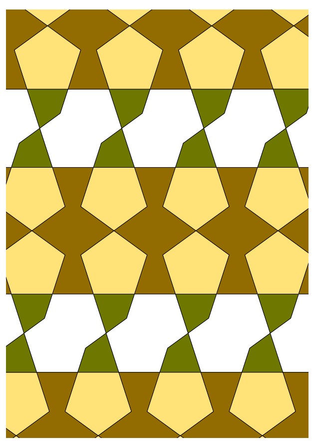

Figure 8: 2222

Figure 8 has four different gyrations of order 2: the centre of the brown polygon, where two brown polygons meet, the centre of the white polygon and where two white polygons meet.

2.3 Miracles and Wonders

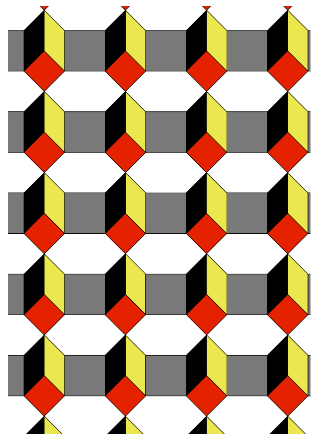

Figure 9: **

In Figure 9, we have two vertical mirror lines through the grey and red squares. This case is different from the mirror line examples previously since the area between the mirror lines is not finite.

We have now considered the symmetries which involve *only* mirrors or *only* gyrations. We have eight more cases.

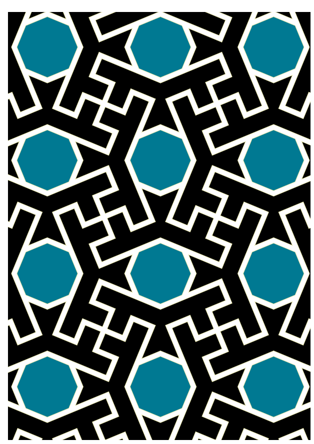

Figure 10: 4*2

In Figure 10, we have mirror lines vertical and horizontally through the octagon. We can also gyrate 180 degrees round the centre of the octagon. A gyration of order four is present about the point at which four black polygons meet. The notation shows all these three properties.

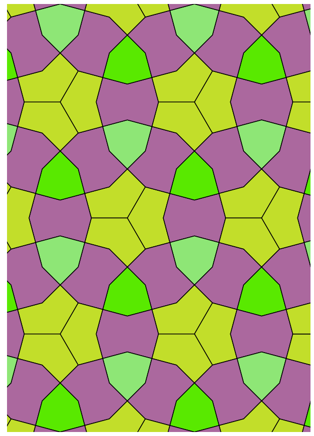

Figure 11: 3*3

Figure 11 has a gyration of order 3 round the green 6-sided polygon, a mirror line by extending the lines through the 6-pointed star, and a gyration round the centre of the 6-pointed star.

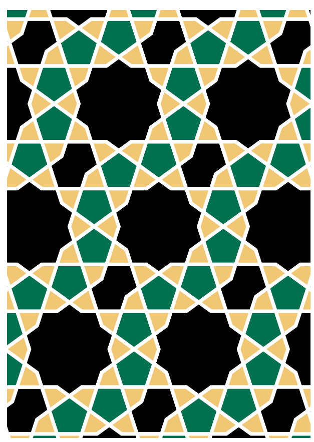

Figure 12: 2*22

Figure 12 mirror lines along the major and minor axes of the black 8-sided figure. There is also a gyration of order 2 round the point midway between the two large black stars.

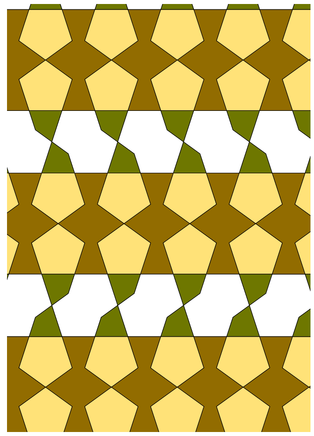

Figure 13: 22*

Figure 13 has a (horizontal) mirror line along the middle of the brown polygons. The pattern can be rotated 180 degrees about the middle of the white 8-sided polygon and also at the point where the two white polygons meet.

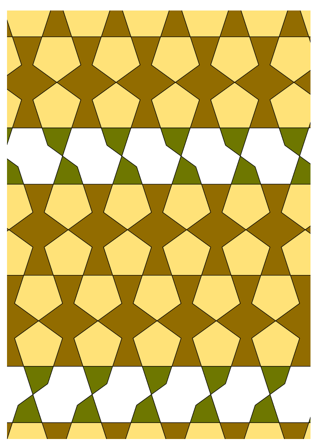

Figure 14: 22×

In Figure 14 we have two rotations by 180 degrees; one about the centre of the white polygon and the other where two white polygons meet. But there is another symmetry: consider a brown polygon. It can be moved along horizontally and then reflected in the line between the brown polygons. This symmetry operation is called a *miracle* and is denoted by ×. Hence the signature for the pattern above.

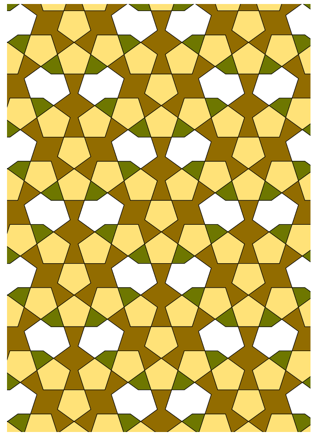

Figure 15: *×

In Figure 15 we have mirror lines through the vertical brown polygons. If one considers one of the non-vertical brown polygons, they can be moved vertical down to the next line of white polygons and flipped over left-to-right. The last operation of clearly the miracle be met before giving the signature above.

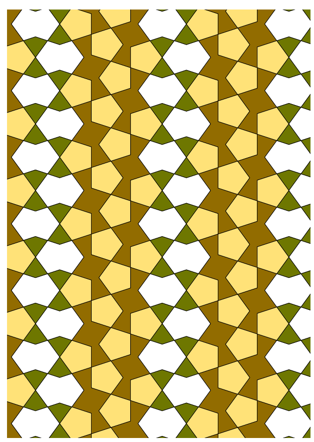

Figure 16: ××

In Figure 16 we have no mirror lines. However, the brown polygon can be moved both up and down and flipped left-to-right. In other words, we have two miracles. Hence the signature for this pattern.

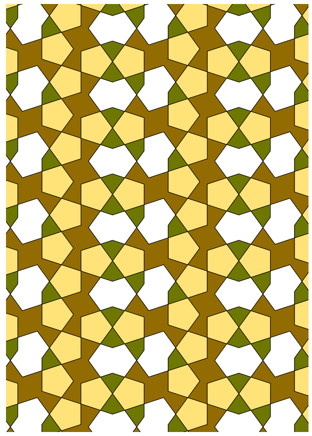

Figure 17: O

In Figure 17 we have no mirror lines, rotations or miracles and is called a *wonder*.

3 The new notation — frieze groups

The symmetry groups of frieze patterns are being introduced. These are illustrated in chapter 5 of [1]. Examples will be added here when they appear on the web site.

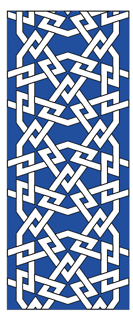

Figure 18: $*22\infty$

In Figure 18 we have a frieze pattern $*22\infty$ (*pma2*).

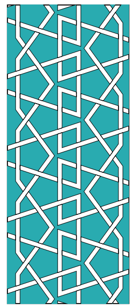

Figure 19: 2*∞

In Figure 19 we have a frieze pattern 2*∞ (*pmm2*).

4 The new notation — rosette groups

The symmetry groups of frieze patterns are being introduced. These are briefly mentioned on page 10 of [1]. Examples will be added here when they appear on the web site.

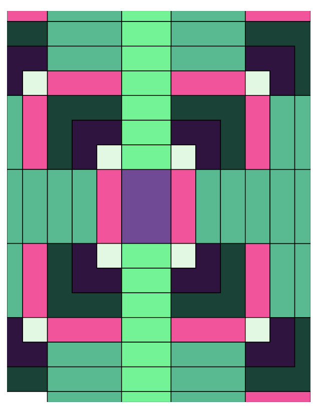

Figure 20: *2•

In Figure 20 we have a rosette pattern *2• (*d2*).

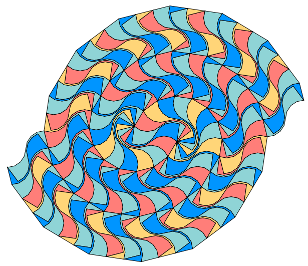

Figure 21: 2•

In Figure 21 we have a rosette pattern 2• (*c2*).

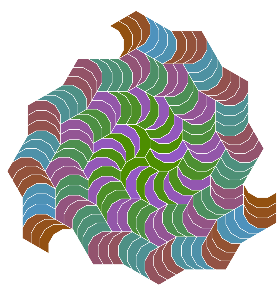

Figure 22: 3•

In Figure 22 we have a rosette pattern 3• (*c3*).

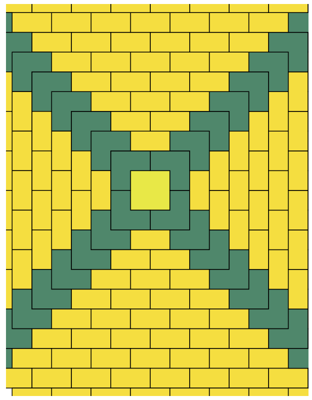

Figure 23: *4•

In Figure 23 we have a rosette pattern *4• (*d4*).

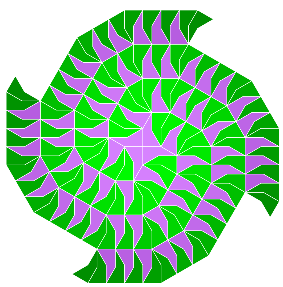

Figure 24: 4•

In Figure 24 we have a rosette pattern 4• (*c4*).

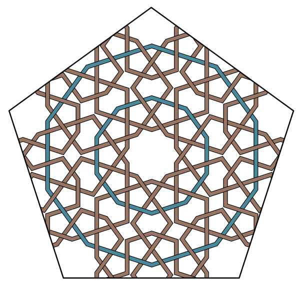

Figure 25: *5•

In Figure 25 we have a rosette pattern *5• ($d5$).

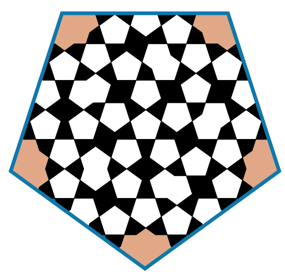

Figure 26: 5•

In Figure 26 we have a rosette pattern 5• ($c5$).

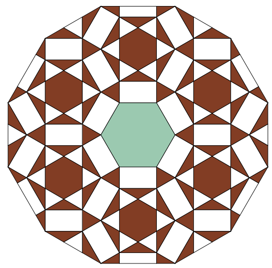

Figure 27: *6•

In Figure 27 we have a rosette pattern *6• ($d6$).

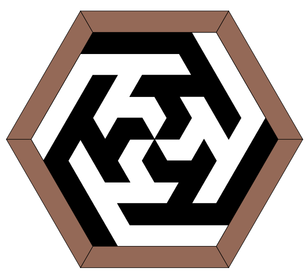

Figure 28: 6•

In Figure 28 we have a rosette pattern 6• ($c6$).

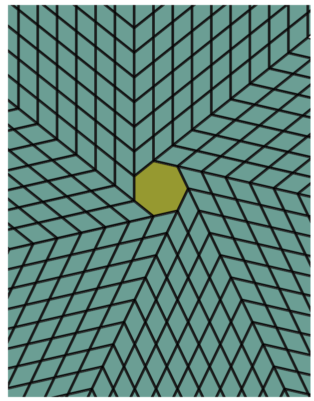

Figure 29: 7•

In Figure 29 we have a rosette pattern 7• ($c7$).

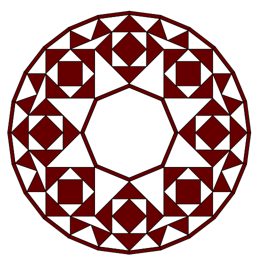

Figure 30: *8•

In Figure 30 we have a rosette pattern *8• ($d8$).

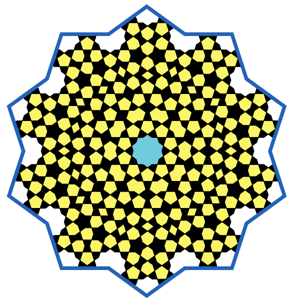

Figure 31: *d10•

In Figure 31 we have a rosette pattern *10• (*d10*).

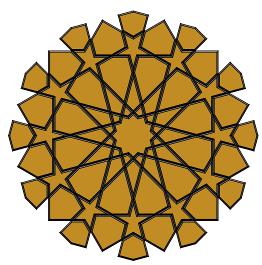

Figure 32: *12•

In Figure 32 we have a rosette pattern *12• (*d12*).

5 Notations compared

The signature used in the new notation is compared which the conventional notation. The frieze groups are added at the bottom follow by the start of the rosette groups.

*× (*cm*)
2*22 (*cmm*)
O (*p1*)
2222 (*p2*)
333 (*p3*)
3*3 (*p3lm*)
*333 (*p3ml*)
442 (*p4*)
4*2 (*p4g*)
*442 (*p4m*)
632 (*p6*)
*632 (*p6m*)
×× (*pg*)
22× (*pgg*)
** (*pm*)
22* (*pmg*)
*2222 (*pmm*)
∞∞ (*p111*)
∞× (*p1a1*)
∞* (*p1m1*)
*∞∞ (*pm11*)
22∞ (*p112*)
*22∞ (*pma2*)
2*∞ (*pmm2*)
*2• (*d2*)
2• (*c2*)
etc

6 Conclusions

This explanation can be used to determine the signature of any repeating pattern. However, a fuller explanation is available in [1]. Note that the area of many of the more complex patterns the displayed on this web site use a scale which implies the edge needs to be inspected carefully to determine how it repeats.

Thanks to Chaim Goodman-Strauss for encouragement in this enterprise.

References

[1] J H Conway, H Burgiel and C Goodman-Strauss. *The symmetry of things.* A K Peters Ltd. 2008. ISBN 978 1 56881 220 5

[2] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman & Co., New York, NY, 1987.

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